Absorption strength in absorbing chaotic cavities

G. Báez,[1] M. Martínez-Mares,[2] and R. A. Méndez-Sánchez[3]

[1] *Área de Física Teórica y Materia Condensada, Universidad Autónoma Metropolitana-Azcapotzalco, Apartado Postal 21-267, 04000 Distrito Federal, Mexico*

[2] *Departamento de Física, Universidad Autónoma Metropolitana-Iztapalapa, Apartado Postal 55-534, 09340 Distrito Federal, Mexico*

[3] *Instituto de Ciencias Físicas, Universidad Nacional Autónoma de México, Apartado Postal 48-3, 62210 Cuernavaca Morelos, Mexico*

(Received 1 October 2007; revised manuscript received 14 March 2008; published 9 September 2008)

We derive an exact formula to calculate the absorption strength in absorbing chaotic systems such as microwave cavities or acoustic resonators. The formula allows us to estimate the absorption strength as a function of the averaged reflection coefficient and the real coupling parameter. We also define the weak and strong absorption regimes in terms of the coupling parameter and the absorption strength.

DOI: 10.1103/PhysRevE.78.036208 PACS number(s): 05.45.−a, 73.23.−b, 03.65.Nk, 42.25.Bs

I. INTRODUCTION

There is increasing attention to wave systems such as cavities with internal losses or absorption, whose ray dynamics is fully chaotic [1]. Although in some experimental cases the absorption in these systems is small [2–4], in many cases the losses are unavoidable [5,6]. The losses should be taken into account in models of microwave cavities at room temperature [2,7–9], in microwave networks [10,11], in elastic and acoustic systems [9,12–18], in optical systems [19,20], and in other applications [16,21]. Furthermore, the effect of losses on chaotic cavities has originated many theoretical models [22–34] (for a recent review see [1]) and experimental works [1,5,6,9,10,24,35–44].

One important question in real absorbing systems is related to the quantification of the losses suffered in a given experimental situation. In some experiments the absorption strength γ can partially be controlled by introducing additional antennas [38] or by including absorbing materials inside the system [6,24]. However, γ cannot be controlled in most experimental cases.

The absorption strength is usually estimated from experimental data with no theoretical basis [6,24,35]. For example, in the setup of Refs. [6,24] (a single-mode port microwave resonator), the value of γ in the theory was fitted to reproduce the experimental value of the average of the reflection coefficient. This average is a monotonically decreasing function of γ. A similar procedure was used in Ref. [35] but with the transmission coefficient T in a two-channel case. In that case, γ was chosen to be the value for which the theoretical distribution of T fits the experimental data. The value of γ can also be estimated by the Fourier transform of the autocorrelation function [44].

In current experiments, the coupling between the waveguide (antenna) and the cavity is not perfect. This gives rise to direct reflections of the wave just before it enters the cavity. This circumstance makes difficult the calculation of the absorption strength γ. A precise procedure to determine γ when the coupling is imperfect would be useful.

Here we present a semianalytical formula to calculate γ which takes into account the coupling for the one-channel case. This formula is helpful in experiments with microwave networks and one-port cavities. In Sec. II we summarize the existing theory used to describe the one-channel-scattering process through chaotic cavities with losses that take into account imperfect coupling. In the same section some known results, valid when time reversal invariance (TRI) is present, are extended to the case when TRI is absent. Our main result is obtained in Sec. III where we calculate the average of the reflection coefficient in terms of γ and the coupling intensity T_a. This relation is inverted numerically to obtain γ in terms of the averaged reflection coefficient. We present our conclusions in Sec. IV.

II. THE $\widetilde{S}$ MATRIX AND ITS DISTRIBUTION

The scattering of waves in a cavity with losses perfectly coupled to the exterior by a single-channel waveguide can be described by a 1×1 scattering matrix $\widetilde{S}_0$ [see Fig. 1(a)]. Due to the presence of losses, the matrix $\widetilde{S}_0$ is subunitary:

$$\widetilde{S}_0\widetilde{S}_0^{\dagger} < 1, \tag{2.1}$$

but this matrix can be parametrized as

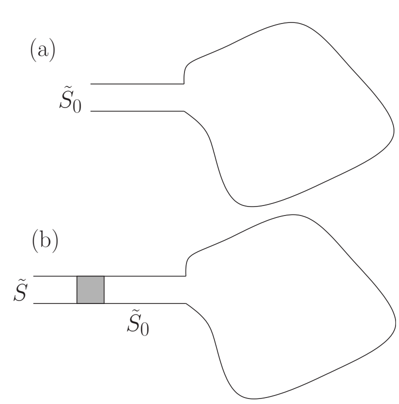

FIG. 1. Sketch of a flat chaotic cavity. In (a) $\widetilde{S}_0$ describes the scattering of the cavity with perfect coupling of the antenna represented as a flat waveguide. In (b) $\widetilde{S}$ describes the scattering through the system cavity plus a barrier that models imperfect coupling.

$$\tilde{S}_0 = \sqrt{R_0} e^{i\theta_0}, \tag{2.2}$$

where R_0 is the reflection coefficient and the phase θ_0 is twice the phase shift (except for an additive constant).

When the classical dynamics of the cavity is chaotic, $\tilde{S}_0$ can be modeled by a random matrix whose elements are chosen following a certain probability distribution. The probability distribution of $\tilde{S}_0$ is assumed to be

$$dP_0^{(\beta)}(\tilde{S}_0) = p_0^{(\beta)}(R_0) dR_0 \frac{d\theta_0}{2\pi}, \tag{2.3}$$

where $\beta=1$ denotes the presence of time reversal symmetry and $\beta=2$ denotes the absence of the same symmetry [24,29]. Note that θ_0 is uniformly distributed between 0 and 2π, while R_0 is distributed according to $p_0^{(\beta)}(R_0)$. For $\beta=1$ we know that [34]

$$p_0^{(1)}(R_0) = \frac{2}{(1-R_0)^2} P_0^{(1)}\left(\frac{1+R_0}{1-R_0}\right), \tag{2.4}$$

where $P_0^{(1)}(x)$, with $x=(1+R_0)/(1-R_0)$ which allows us to calculate the integrated probability distribution

$$W_1(x) = \int_x^\infty dx' P_0^{(1)}(x'). \tag{2.5}$$

We note that $W_1(x)$ is a positive monotonically decaying function, which is explicitly given by [34]

$$W_1(x) = \frac{x+1}{4\pi}[f_1(w)g_2(w) + f_2(w)g_1(w) + h_1(w)j_2(w) + h_2(w)j_1(w)]_{w=(x-1)/2}, \tag{2.6}$$

with

$$f_q(w) = \int_{l_q}^{u_q} dt \frac{\sqrt{t|t-w|}e^{-\gamma t/2}}{(1+t)^{3/2}}(1-e^{-\gamma}+t^{-1}),$$

$$g_q(w) = \int_{l_q}^{u_q} dt \frac{1}{\sqrt{t|t-w|}} \frac{e^{-\gamma t/2}}{(1+t)^{3/2}},$$

$$h_q(w) = \int_{l_q}^{u_q} dt \frac{\sqrt{|t-w|}e^{-\gamma t/2}}{\sqrt{t(1+t)}}[\gamma + (1-e^{-\gamma})(\gamma t - 2)],$$

$$j_q(w) = \int_{l_q}^{u_q} dt \frac{1}{\sqrt{t|t-w|}} \frac{e^{-\gamma t/2}}{\sqrt{1+t}}. \tag{2.7}$$

Here $q=1,2$, $l_1=w$, $l_2=0$, $u_1=\infty$, and $u_2=w$. Several attempts to interpolate $p_0^{(1)}(R_0)$ between the two well-known limits of strong ($\gamma\to\infty$) and weak ($\gamma\to 0$) absorption have been reported [28,29,34].

The following equations for $p_0^{(\beta)}(R_0)$ extend to $\beta=2$ the results of Eqs. (2.4)–(2.6). We can see that

$$p_0^{(\beta)}(R_0) = \frac{2}{(1-R_0)^2} P_0^{(\beta)}\left(\frac{1+R_0}{1-R_0}\right), \tag{2.8}$$

where $P_0^{(2)}(x)$ is related to $W_2(x)$ as in Eq. (2.5) and

$$W_2(x) = \frac{1}{2} e^{-\gamma x/2}[e^{\gamma/2}(x+1) - e^{-\gamma/2}(x-1)]. \tag{2.9}$$

From Eq. (2.8) we obtain the well-known result for $\beta=2$ [33],

$$p_0^{(2)}(R_0) = \frac{e^{-\gamma/(1-R_0)}}{(1-R_0)^3}[\gamma(e^\gamma - 1) + (1+\gamma - e^\gamma)(1-R_0)]. \tag{2.10}$$

We focus now on the scattering system when both losses and imperfect coupling are present. The imperfect coupling can be modeled theoretically by adding a barrier in the waveguide at the entrance of the cavity [see Fig. 1(b)]. The scattering matrix that describes the cavity in this case will be denoted by $\tilde{S}$ and, as before, it can be parametrized in terms of the reflection coefficient R and its phase θ as

$$\tilde{S} = \sqrt{R} e^{i\theta}. \tag{2.11}$$

This new matrix is connected with $\tilde{S}_0$ by means of the transformation [24]

$$\tilde{S}(\tilde{S}_0) = -\sqrt{1-T_a} + \sqrt{T_a}\frac{1}{1-\tilde{S}_0\sqrt{1-T_a}}\tilde{S}_0\sqrt{T_a}, \tag{2.12}$$

where T_a represents the imperfect coupling between the antenna and the cavity. For perfect coupling $T_a=1$ and then $\tilde{S}$ becomes equal to $\tilde{S}_0$. For no coupling we have $\tilde{S}=-1$ for $T_a=0$. The two terms on the right-hand side of the last transformation are the fixed and fluctuating parts of $\tilde{S}$. The first one represents the reflection due to the imperfect coupling at the entrance to the cavity, while the second is the contribution to $\tilde{S}$ of the multiple scattering in the cavity. Therefore, Eq. (2.12) also can be written as

$$\tilde{S} = \langle \tilde{S} \rangle + \tilde{S}_{\text{fluc}}, \tag{2.13}$$

where $\langle \tilde{S} \rangle = -\sqrt{1-T_a}$ is the average of $\tilde{S}$. In general $\langle \tilde{S} \rangle$, known as the *optical scattering matrix*, is a measure of the prompt responses in the system (imperfect coupling) due to direct processes. In our case an imperfect coupling gives rise to direct reflections. The coupling between the antenna and the cavity can be quantified by [5]

$$T_a = 1 - |\langle \tilde{S} \rangle|^2, \tag{2.14}$$

which can be obtained from the experimental data since $\langle \tilde{S} \rangle$ is the average over different realizations (or frequencies) of the measured $\tilde{S}$, including phases. The probability distribution of $\tilde{S}$ is given by [24]

$$dP_{\langle \tilde{S} \rangle}^{(\beta)}(\tilde{S}) = p_{\langle \tilde{S} \rangle}^{(\beta)}(\tilde{S}) dR \frac{d\theta}{2\pi}, \tag{2.15}$$

where

$$p_{\langle \tilde{S} \rangle}^{(\beta)}(\tilde{S}) = \left(\frac{1-\langle \tilde{S} \rangle^2}{|1-\tilde{S}\langle \tilde{S} \rangle|^2} \right)^2 p_0^{(\beta)}[R_0(\tilde{S})]. \qquad (2.16)$$

Here $p_0^{(\beta)}[R_0(\tilde{S})]$ is given by Eq. (2.4). Note that for perfect coupling ($\langle \tilde{S} \rangle=0$ or $T_a=1$), $\tilde{S}$ reduces to $\tilde{S}_0$ and $p_{\langle \tilde{S} \rangle}^{(\beta)}(\tilde{S})$ becomes in $p_0^{(\beta)}(\tilde{S}_0)$. The quadratic term in the large parentheses in Eq. (2.16) is the Jacobian of the transformation [see Eq. (2.12)]; this result was recently generalized to the case of N channels [30].

In the next section we use the formalism presented here in order to calculate the average $\langle R \rangle_\beta$ as a function of the absorption strength γ and the antenna coupling T_a. This relation can be inverted (at least numerically) to give γ as a function of $\langle R \rangle_\beta$ and T_a.

III. THE ABSORPTION STRENGTH γ AS A FUNCTION OF $\langle R \rangle_\beta$ AND T_a

The average $\langle R \rangle_\beta$ can be calculated using directly Eq. (2.15). However, it is necessary to write R_0 as a function of R and θ [see Eq. (2.16)], substituting the resulting expression in the corresponding distribution $p_0^{(\beta)}(R_0)$ above; then $\langle R \rangle_\beta$ should be integrated with respect to R and θ. But instead of doing this long processes, we prefer to calculate $\langle R \rangle_\beta$ using Eq. (2.3) as

$$\langle R \rangle_\beta = \int_0^{2\pi} \frac{d\theta_0}{2\pi} \int_0^1 dR_0 R(R_0,\theta_0) p_0^{(\beta)}(R_0), \qquad (3.1)$$

where we need to write R as a function of R_0 and θ_0. Using Eq. (2.12) and $\langle \tilde{S} \rangle = -\sqrt{1-T_a}$ (the phase in the single-mode case is not needed) we arrive at

$$R = \frac{R_0 + (1-T_a) - 2\sqrt{1-T_a}\sqrt{R_0}\cos\theta_0}{1+(1-T_a)R_0 - 2\sqrt{1-T_a}\sqrt{R_0}\cos\theta_0}. \qquad (3.2)$$

Substituting this expression for R into Eq. (3.1) and integrating with respect to θ_0, we get

$$\langle R \rangle_\beta = 1 - T_a \int_0^1 \frac{1-R_0}{1-(1-T_a)R_0} p_0^{(\beta)}(R_0) dR_0, \qquad (3.3)$$

where we have used that $p_0^{(\beta)}(R_0)$ is normalized to unity. The dependence of $\langle R \rangle_\beta$ on γ comes through $p_0^{(\beta)}(R_0)$ as can be seen in Eqs. (2.4)–(2.7) and (2.10). This equation is our main result, providing an exact expression for the average of R as a function of γ and T_a. This expression could be useful for experimentalists. However, we show below that Eq. (3.3) can be written in two more practical forms [see Eqs. (3.4) and (3.5) below], especially for the case $\beta=1$.

Let us check the two limits: the no-coupling and the perfect coupling limits. For $T_a=0$, $\langle R \rangle_\beta=1$, which is compatible with the argument that the waveguide is blocked, the wave never enters the cavity, and hence there are no losses. At the opposite limit, $T_a=1$ leads to $\langle R \rangle_\beta = \langle R_0 \rangle_\beta$, which is expected again because the coupling is perfect and such that $\tilde{S}=\tilde{S}_0$.

When $0<T_a<1$, the integral in Eq. (3.3) can be done numerically. For $\beta=2$ we directly substitute Eq. (2.10) into

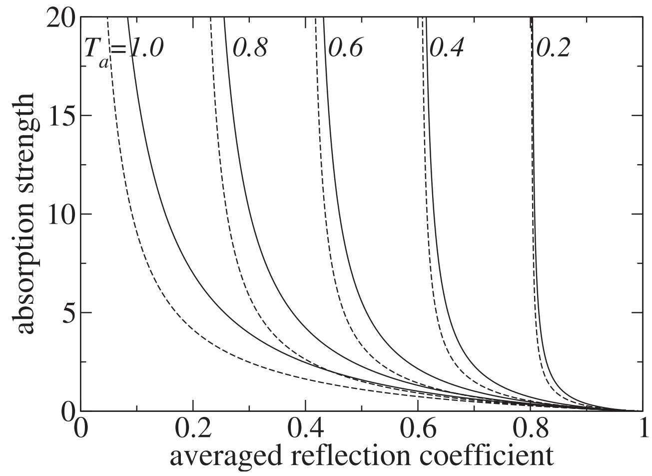

FIG. 2. The average of the reflection coefficient $\langle R \rangle_\beta$ is a monotonically decreasing function of the absorption strength γ. We show γ as a function of $\langle R \rangle_\beta$ for several values of the coupling T_a: from left to right $T_a=1.0, 0.8, 0.6, 0.4, 0.2$. Continuous lines correspond to $\beta=1$ and dashed lines to $\beta=2$. To a given value of $\langle R \rangle_\beta$ corresponds an infinite number of γ values, and vice versa.

Eq. (3.3) and integrate numerically with respect to R_0. The calculation for $\beta=1$ is more complicated. However, a simpler formula for $\langle R \rangle_\beta$ can be obtained by integrating Eq. (3.3) by parts, to get (see Appendix A)

$$\langle R \rangle_\beta = 1 - T_a + 2T_a^2 \int_1^\infty \frac{W_\beta(x)}{[x+1-(1-T_a)(x-1)]^2} dx. \qquad (3.4)$$

The remaining integrations can be done numerically. The advantage of this equation compared with Eq. (3.3) is that it is numerically more stable, especially for $\beta=1$.

In Fig. 2 we show the numerical results of the last equation for $\beta=1$ and 2. This is done for several values of T_a. Instead of giving $\langle R \rangle_\beta$ as a function of γ, we plot γ as a function of $\langle R \rangle_\beta$, which could be more useful to experimentalists. We observe that $\langle R \rangle_\beta$ decays with γ for T_a fixed or with T_a for γ fixed. This means that the coupling has an effect on the averaged reflection coefficient similar to that of the losses. In this sense we say that the coupling mimics the absorption, and vice versa [6]. This can be clarified by writing the argument of the integral of Eq. (3.3) in powers of $(1-T_a)R_0$ and performing the integral term by term. The result is an exact useful expression for $\langle R \rangle_\beta$, namely,

$$\langle R \rangle_\beta = 1 - T_a + T_a^2 \sum_{n=1}^{\infty} (1-T_a)^{(n-1)} \langle R_0^n \rangle_\beta, \qquad (3.5)$$

where

$$\langle R_0^n \rangle_\beta = \int_0^1 R_0^n p_0^{(\beta)}(R_0) dR_0. \qquad (3.6)$$

The first two terms $(1-T_a)$ in Eq. (3.5) give the direct reflection due to the coupling at the entrance to the cavity, while the remaining terms are due to reflections after the multiple scattering. The quantity T_a^2 comes from the probability of the transmission to cross the barrier twice (entering and leaving the cavity). The sum represents the multiple reflections between the cavity walls and the barrier. Again, we check the two limits of no coupling and perfect coupling: $T_a=0$ implies $\langle R\rangle_\beta=1$, while $\langle R\rangle_\beta=\langle R_0\rangle_\beta$ for $T_a=1$. We usually obtain both parameters T_a and γ from experimental measurements. These parameters are needed in the theoretical predictions. Once T_a is calculated from Eq. (2.14), γ is obtained from $\langle R\rangle_\beta$, Eqs. (3.3) and (3.4), which is also obtained from experimental data. Our formula allows us to distinguish which part of the wave is reflected and which part is lost by absorption.

Equations (3.5) and (3.4) are analogous to the transformation (2.13) where $\widetilde{S}=\langle\widetilde{S}\rangle+\widetilde{S}_{\rm fluc}$, but for the averaged reflection coefficient. A formula is given in Refs. [46,47] for the delay time in which the prompt and delayed responses are separated.

A. Strong absorption regime

In this limit it is usually understood that $\gamma\gg1$. Here we will show that the criterion for strong absorption, when direct reflections are also present, is $\gamma\gg T_a$. Notice that the old criterion is included in the new one since $T_a\in[0,1]$.

The probability density distribution of R_0 in this limit reduces to [29,33]

$$p_0^{(\beta)}(R_0)=\alpha_\beta\frac{e^{-\alpha_\beta R_0/(1-R_0)}}{(1-R_0)^{2+\beta/2}}, \tag{3.7}$$

where we have defined $\alpha_\beta=\gamma\beta/2$. Substituting Eq. (3.7) into Eq. (3.6) we can show that (see Appendix B 1)

$$\langle R_0^n\rangle_\beta\approx\frac{n!}{\alpha_\beta^n}\to0\quad\text{as }\alpha_\beta\to\infty. \tag{3.8}$$

This result is consistent with the result $\langle R_0\rangle_\beta=1/\alpha_\beta$ of Ref. [22]. Then Eq. (3.5) gives

$$\langle R\rangle_\beta\approx1-T_a. \tag{3.9}$$

This means that, once the wave enters a cavity with strong absorption, it never gets back since the only reflection happens at the entrance of the cavity (barrier).

In Fig. 3 we compare the exact result of Eq. (3.4) with the strong absorption regime of Eq. (3.9), which corresponds to a horizontal line. This is done for two different values of T_a. As can be seen in this figure, the exact solution for $T_a=0.2$ is close to the asymptotic behavior $1-T_a$ when $\gamma\approx4$. This is in contradiction with the criterion $\gamma\gg1$. In this case we can see that a criterion for strong absorption is $\gamma\gg T_a$.

B. Weak absorption regime

The weak absorption regime is usually defined by $\gamma\ll1$. Here we will show that a generalized criterion for weak absorption, when direct reflections are present, is $\gamma\ll T_a$. Notice that now the old criterion is satisfied when the generalized criterion is valid since $T_a\le1$.

In the weak absorption regime [33]

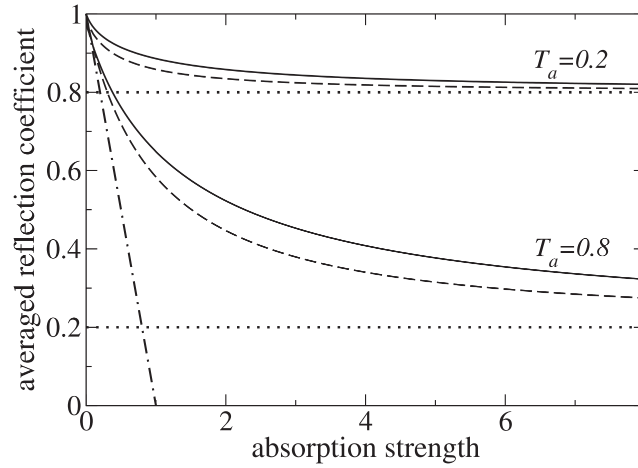

FIG. 3. Averaged reflection coefficient $\langle R\rangle_\beta$ as a function of the absorption strength γ for two values of T_a. The continuous and dashed lines correspond to $\beta=1$ and 2, respectively. The dotted lines yield the strong absorption limit, Eq. (3.9). The weak absorption limit, Eq. (3.15), is given by the dash-dotted line.

$$p_0^{(\beta)}(R_0)=\frac{\alpha_\beta^{1+\beta/2}}{\Gamma(1+\beta/2)}\frac{e^{-\alpha_\beta/(1-R_0)}}{(1-R_0)^{2+\beta/2}}, \tag{3.10}$$

where $\Gamma(x)$ is the Gamma function [45]. Substituting this result into Eq. (3.6), we get (see Appendix B 2)

$$\langle R_0^n\rangle_\beta\approx1-\frac{2n}{\beta}\alpha_\beta=1-n\gamma. \tag{3.11}$$

Inserting this result in Eq. (3.5), we obtain

$$\langle R\rangle_\beta\approx(1-T_a)+\frac{T_a^2}{1-T_a}\sum_{n=1}^{\infty}(1-T_a)^n-\gamma T_a^2\sum_{n=1}^{\infty}n(1-T_a)^{n-1}, \tag{3.12}$$

which can be written as

$$\langle R\rangle_\beta\approx1-T_a-\frac{T_a^2}{1-T_a}+\frac{T_a^2}{1-T_a}\sum_{n=0}^{\infty}(1-T_a)^n-\gamma T_a^2\sum_{n=1}^{\infty}n(1-T_a)^{n-1}. \tag{3.13}$$

The fourth term on the right-hand side is just a geometric series and can be summed. The result is

$$\langle R\rangle_\beta\approx(1-T_a)+T_a\left(1-\gamma T_a\sum_{n=1}^{\infty}n(1-T_a)^{n-1}\right). \tag{3.14}$$

Here, we note that the first term $(1-T_a)$ is the reflected part of the incident wave at the entrance of the cavity and the second is due to multiple reflections within the cavity. The sum inside the second term is the derivative with respect to $(1-T_a)$ of the infinite geometric series whose sum is just $1/T_a^2$. This leads to

$$\langle R\rangle_\beta \approx (1-T_a)+T_a-\gamma = 1-\gamma. \tag{3.15}$$

In this limit a small part of the wave that enters into the cavity is lost by absorption and the rest contributes to the reflection. Therefore, the reflection coefficient $\langle R\rangle_\beta$ is slightly less than unity; it contains a reflected part due to the coupling parameter and a part that is not lost by absorption. Notice that the first term coming from the multiple scattering, T_a, cancels the $-T_a$ that comes from the direct reflection. Therefore T_a does not appear explicitly in the final expression for $\langle R\rangle_\beta$, to first order in γ.

In Fig. 3 we compare also the average of R in the weak absorption limit with the exact result of Eq. (3.4) as a function of the absorption strength γ for two values of T_a. One can see in this figure that, in the case of $T_a=0.2$ and, say, $\gamma=0.1$, the asymptotic behavior is not reached. This is in contradiction with the criterion $\gamma \ll 1$. In this case the appropriate criterion is $\gamma \ll T_a$.

Equations (3.9) and (3.15) represent the average of R in the two limits of strong and weak absorption, respectively. Notice that they are independent of the symmetry β. Let us emphasize that our criterion for strong or weak absorption, when the coupling is not perfect, becomes $\gamma \gg T_a$ or $\gamma \ll T_a$. This agrees with $\gamma \gg 1$ or $\gamma \ll 1$, respectively, for strong or weak absorption with perfect coupling.

IV. CONCLUSIONS

We have presented a semianalytical formula to calculate the absorption strength γ due to losses in a chaotic cavity which takes into account the imperfect coupling of the single-channel port. This formula could be useful for experiments with microwave networks and one-port cavities where an accurate value of γ is needed. We have shown that the real imperfect coupling and the absorption have a similar effect on the scattering properties. Also, a precise definition of the strong and weak absorption regimes when imperfect coupling is present was introduced.

ACKNOWLEDGMENTS

The authors acknowledge financial support from DGAPA-UNAM, México, under Projects No. IN118805 and No. IN111308. We are indebted to J. Flores, José A. Heras, and A. L. Salas-Brito for a careful reading of the manuscript.

APPENDIX A: CALCULATION OF EQ. (3.4)

We start with the substitution of $p_0^{(\beta)}(R_0)$ of Eq. (2.8) into Eq. (3.3). With an appropriate change of variables $x=(1+R_0)/(1-R_0)$, the result can be written as

$$\langle R\rangle_\beta = 1-2T_a I_\beta, \tag{A1}$$

where

$$I_\beta = \int_1^\infty A(x)P_0^{(\beta)}(x)dx, \tag{A2}$$

with $P_0^{(\beta)}(x)$ giving rise to $W_\beta(x)$ as in Eq. (2.5) and

$$A(x) = \frac{1}{x+1-(1-T_a)(x-1)}. \tag{A3}$$

We integrate by parts, identifying

$$u = A(x), \quad du = \frac{dA(x)}{dx}dx, \tag{A4}$$

$$v = -W_\beta(x), \quad dv = P_0^{(\beta)}(x)dx. \tag{A5}$$

The result of the integration is

$$I_\beta = -A(x)W_\beta(x)|_{x=1}^{\infty} + \int_1^\infty W_\beta(x)\frac{dA(x)}{dx}dx. \tag{A6}$$

For $\beta=2$, Eqs. (2.9) and (A3) give

$$A(x)W_2(x)|_{x=1}^{\infty} = \frac{1}{2}. \tag{A7}$$

We show below that the same result is valid for $\beta=1$.

First, we evaluate $A(x)W_1(x)$ at $x=\infty$ or $w=\infty$, using Eqs. (2.6) and (A3). For instance, we consider the term $f_1(w)g_2(w)$ appearing in Eq. (2.6). Defining $y=t/w$, we can write

$$\begin{aligned} f_1(w)g_2(w)|_{w=\infty} &= (1-e^{-\gamma})\lim_{w\to\infty}\int_1^\infty \sqrt{\frac{y(y-1)}{(1+wy)^3}}we^{-\gamma wy/2}dy \\ &\quad \times \int_0^1 \frac{we^{-\gamma wy/2}dy}{\sqrt{y(1-y)(1+wy)^3}} \\ &\quad + \lim_{w\to\infty}\int_1^\infty \sqrt{\frac{y-1}{y(1+wy)^3}}we^{-\gamma wy/2}dy \\ &\quad \times \int_0^1 \frac{e^{-\gamma wy/2}}{\sqrt{y(1-y)(1+wy)^3}}dy. \end{aligned} \tag{A8}$$

Here, we can use a definition of the Dirac δ function, namely,

$$\lim_{w\to\infty} we^{-\gamma wy/2} = \frac{2}{\gamma}\delta(y). \tag{A9}$$

The interval of integration in Eq. (A8) does not include the argument of $\delta(y)$. As a consequence $f_1(w)g_2(w)|_{w=\infty}=0$. In a similar way it can be shown that the remaining terms in Eq. (2.6) give zero when evaluated at $w=\infty$. Then,

$$A(x)W_1(x)|_{x=\infty} = 0. \tag{A10}$$

We now consider the evaluation of $A(x)W_1(x)$ at $x=1$ or $w=0$. From Eqs. (2.7) we see that the first term in $W_1(x)$ gives

$$\begin{aligned} f_1(w)g_2(w)|_{w=0} &= \lim_{w\to 0}\int_0^1 \frac{e^{-\gamma wy/2}}{\sqrt{y(1-y)(1+wy)^3}}dy \\ &\quad \times \int_w^\infty \sqrt{\frac{t(t-w)}{(1+t)^3}}e^{-\gamma t/2}\left(1-e^{-\gamma}+\frac{1}{t}\right)dt, \end{aligned} \tag{A11}$$

which reduces to

$$f_1(w)g_2(w)|_{w=0} = \pi\int_0^\infty \frac{te^{-\gamma t/2}}{(1+t)^{3/2}}\left(1-e^{-\gamma}+\frac{1}{t}\right)dt. \tag{A12}$$

Similarly, the third term in $W_1(x)$ gives

$$h_1(w)j_2(w)|_{w=0} = \pi\int_0^\infty \frac{e^{-\gamma t/2}}{\sqrt{1+t}}[\gamma+(1-e^{-\gamma})(\gamma t-2)]dt, \tag{A13}$$

while the second and the fourth terms in $W_1(x)$ are zero when evaluated at $x=1$. Let us consider the second term only:

$$\begin{aligned} f_2(w)g_1(w)|_{w=0} = \lim_{w\to 0}\int_0^1 &\sqrt{\frac{y(1-y)}{(1+wy)^3}}w^2 \\ &\times e^{-\gamma wy/2}\left(1-e^{-\gamma}+\frac{1}{wy}\right)dy \\ &\times\int_w^\infty \frac{e^{-\gamma t/2}}{\sqrt{t(t-w)(1+t)^3}}dt, \end{aligned} \tag{A14}$$

which gives zero.

From Eqs. (A12), (A13), and (A3) we get

$$\begin{aligned} A(x)W_1(x)|_{x=1} = \frac{1}{4}(1-e^{-\gamma})&\int_0^\infty \frac{te^{-\gamma t/2}}{(1+t)^{3/2}} + \int_0^\infty \frac{e^{-\gamma t/2}}{(1+t)^{3/2}} \\ &+(\gamma-2+2e^{-\gamma})\int_0^\infty \frac{e^{-\gamma t/2}}{(1+t)^{1/2}} \\ &+\gamma(1-e^{-\gamma})\int_0^\infty \frac{te^{-\gamma t/2}}{(1+t)^{1/2}}. \end{aligned} \tag{A15}$$

Integrating by parts, we can establish the following relations:

$$\int_0^\infty \frac{e^{-\gamma t/2}}{(1+t)^{1/2}}dt = \frac{2}{\gamma}-\frac{1}{\gamma}\int_0^\infty \frac{e^{-\gamma t/2}}{(1+t)^{3/2}}dt, \tag{A16}$$

$$\begin{aligned} \int_0^\infty \frac{te^{-\gamma t/2}}{(1+t)^{1/2}}dt = \left(\frac{2}{\gamma}\right)^2 &- \frac{1}{\gamma}\int_0^\infty \frac{te^{-\gamma t/2}}{(1+t)^{3/2}}dt \\ &-\frac{2}{\gamma^2}\int_0^\infty \frac{e^{-\gamma t/2}}{(1+t)^{3/2}}dt. \end{aligned} \tag{A17}$$

Substituting Eqs. (A16) and (A17) into Eq. (A15) we obtain

$$A(x)W_1(x)|_{x=1} = \frac{1}{2}. \tag{A18}$$

Therefore, using Eqs. (A7), (A10), and (A18), Eq. (A6) can be written as

$$I_\beta = \frac{1}{2}+\int_1^\infty W_\beta(x)\frac{dA(x)}{dx}dx. \tag{A19}$$

Finally, Eqs. (A1), (A3), and (A19) yield Eq. (3.4).

APPENDIX B: CALCULATION OF $\langle R_0^n\rangle_\beta$

1. Strong absorption limit

In this limit, Eq. (3.7) can still be simplified to the Rayleigh distribution [22,29]

$$p_0^{(\beta)}(R_0) = \alpha_\beta e^{-\alpha_\beta R_0}, \tag{B1}$$

which is substituted into Eq. (3.6) to obtain

$$\langle R_0^n\rangle_\beta = \alpha_\beta\int_0^1 R_0^n e^{-\alpha_\beta R_0}dR_0 = -e^{-\alpha_\beta}+\frac{n}{\alpha_\beta}\langle R_0^{n-1}\rangle_\beta, \tag{B2}$$

where an integration by parts was done. This expression can be iterated to obtain

$$\langle R_0^n\rangle_\beta = \frac{n!}{\alpha_\beta^n} - e^{-\alpha_\beta}\sum_{m=0}^{n-1}\frac{n!}{(n-m)!\,\alpha_\beta^m}. \tag{B3}$$

The limit $\alpha_\beta\to\infty$ gives Eq. (3.8).

2. Weak absorption limit

We substitute Eq. (3.10) into Eq. (3.6) to obtain

$$\begin{aligned} \langle R_0^n\rangle_\beta &= \frac{\alpha_\beta^{1+\beta/2}}{\Gamma(1+\beta/2)}\int_0^1 R_0^n\frac{e^{-\alpha_\beta/(1-R_0)}}{(1-R_0)^{2+\beta/2}}dR_0 \\ &= \frac{\alpha_\beta^{1+\beta/2}}{\Gamma(1+\beta/2)}\int_1^\infty e^{-\alpha_\beta x}(x-1)^n x^{\beta/2-n}dx, \end{aligned} \tag{B4}$$

where we used the change of variable $x=(1-R_0)^{-1}$. Using the binomial expansion for $(x-1)^n$ we write Eq. (B4) as

$$\langle R_0^n\rangle_\beta = \frac{1}{\Gamma(1+\beta/2)}\sum_{r=0}^n \frac{(-\alpha_\beta)^{n-r}n!}{r!(n-r)!}\Gamma(1+\beta/2+r-n,\alpha), \tag{B5}$$

where $\Gamma(a,x)$ is the incomplete Gamma function [45]. Keeping linear terms in α_β, we arrive at Eq. (3.11). Special care for $\beta=2$ should be taken since the Gamma functions are divergent. However, the result of Eq. (3.11) is valid.

[1] Y. V. Fyodorov, D. V. Savin, and H.-J. Sommers, J. Phys. A **38**, 10731 (2005).
[2] H.-D. Gräf, H. L. Harney, H. Lengeler, C. H. Lewenkopf, C. Rangacharyulu, A. Richter, P. Schardt, and H. A. Weidenmüller, Phys. Rev. Lett. **69**, 1296 (1992).
[3] H. Alt, A. Bäcker, C. Dembowski, H.-D. Gräf, R. Hofferbert, H. Rehfeld, and A. Richter, Phys. Rev. E **58**, 1737 (1998).
[4] L. Collot, V. Lefèvre-Seguin, M. Brune, J. M. Raimond, and S. Haroche, Europhys. Lett. **23**, 327 (1993).
[5] C. H. Lewenkopf, A. Müller, and E. Doron, Phys. Rev. A **45**, 2635 (1992).
[6] R. A. Méndez-Sánchez, U. Kuhl, M. Barth, C. H. Lewenkopf, and H.-J. Stöckmann, Phys. Rev. Lett. **91**, 174102 (2003).
[7] H.-J. Stöckmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990).
[8] S. Sridhar, Phys. Rev. Lett. **67**, 785 (1991).
[9] U. Kuhl, H.-J. Stöckmann, and R. Weaver, J. Phys. A **38**, 10433 (2005).
[10] O. Hul, S. Bauch, P. Pakoński, N. Savytskyy, K. Zyczkowski, and L. Sirko, Phys. Rev. E **69**, 056205 (2004).
[11] O. Hul, O. Tymoshchuk, S. Bauch, P. M. Koch, and L. Sirko, J. Phys. A **38**, 10489 (2005).
[12] E. Arcos, G. Báez, P. Cuatláyol, M. L. H. Prian, R. A. Méndez-Sánchez, and H. Hernández-Saldaña, Am. J. Phys. **66**, 601 (1998).
[13] C. Ellegaard, K. Schaadt, and P. Bertelsen, Phys. Scr., T **90**, 223 (2001).
[14] L. Gutiérrez, A. Díaz-de-Anda, J. Flores, R. A. Méndez-Sánchez, G. Monsivais, and A. Morales, Phys. Rev. Lett. **97**, 114301 (2006).
[15] G. Tanner and N. Søndergaard, J. Phys. A: Math. Theor. **40**, R443 (2007).
[16] R. Blümel, I. H. Davidson, W. P. Reinhardt, H. Lin, and M. Sharnoff, Phys. Rev. A **45**, 2641 (1992).
[17] K. Schaadt and A. Kudrolli, Phys. Rev. E **60**, R3479 (1999).
[18] G. Montaldo, N. Perez N, C. Negreira, and M. Fink, Waves Random Media **17**, 67 (2007).
[19] V. Doya, O. Legrand, and F. Mortessagne, Opt. Lett. **26**, 872 (2001); P. Leproux, V. Doya, D. Pagnoux, F. Mortesagne, and O. Legrand, Opt. Commun. **218**, 249 (2003).
[20] A. Mekis, J. U. Nöckel, G. Chen, A. D. Stone, and R. K. Chang, Phys. Rev. Lett. **75**, 2682 (1995).
[21] A. Kudrolli, M. C. Abraham, and J. P. Gollub, Phys. Rev. E **63**, 026208 (2001).
[22] E. Kogan, P. A. Mello, and He Liqun, Phys. Rev. E **61**, R17 (2000).
[23] M. Martínez-Mares and P. A. Mello, Phys. Rev. E **72**, 026224 (2005).
[24] U. Kuhl, M. Martínez-Mares, R. A. Méndez-Sánchez, and H.-J. Stöckmann, Phys. Rev. Lett. **94**, 144101 (2005).
[25] D. V. Savin, Y. V. Fyodorov, and H.-J. Sommers, Acta Phys. Pol. A **109**, 53 (2006).
[26] S. H. Simon and A. L. Moustakas, Phys. Rev. Lett. **96**, 136805 (2006).
[27] I. Rozhkov and G. Murty, J. Phys. A **38**, 10843 (2005).
[28] D. V. Savin and H.-J. Sommers, Phys. Rev. E **68**, 036211 (2003).
[29] M. Martínez-Mares and R. A. Méndez-Sánchez, J. Phys. A **38**, 010873 (2005).
[30] V. A. Gopar, M. Martínez-Mares, and R. A. Méndez-Sánchez, J. Phys. A: Math. Theor. **41**, 015103 (2008).
[31] V. Domínguez-Rocha, C. Zagoya, and M. Martínez-Mares, Am. J. Phys. **76**, 621 (2008).
[32] D. V. Savin, O. Legrand, and F. Mortessagne, Europhys. Lett. **76**, 774 (2006).
[33] C. W. J. Beenakker and P. W. Brouwer, Physica E (Amsterdam) **9**, 463 (2001).
[34] D. V. Savin, H.-J. Sommers, and Y. V. Fyodorov, JETP Lett. **82**, 544 (2005).
[35] H. Schanze, H.-J. Stöckmann, M. Martínez-Mares, and C. H. Lewenkopf, Phys. Rev. E **71**, 016223 (2005).
[36] S. Hemmady, X. Zheng, E. Ott, T. M. Antonsen, Jr., and S. M. Anlage, Phys. Rev. Lett. **94**, 014102 (2005).
[37] S. Hemmady, X. Zheng, T. M. Antonsen, E. Ott, and S. M. Anlage, Phys. Rev. E **71**, 056215 (2005).
[38] J. Barthelemy, O. Legrand, and F. Mortessagne, Phys. Rev. E **71**, 016205 (2005).
[39] J. Barthelemy, O. Legrand, and F. Mortessagne, Europhys. Lett. **70**, 162 (2005).
[40] X. Zheng, T. M. Antonsen, and E. Ott, Electromagnetics **26**, 3 (2006).
[41] X. Zheng, T. M. Antonsen, and E. Ott, Electromagnetics **26**, 37 (2006).
[42] X. Zheng, S. Hemmady, T. M. Antonsen, S. M. Anlage, and E. Ott, Phys. Rev. E **73**, 046208 (2006).
[43] S. Hemmady, J. Hart, X. Zheng, T. M. Antonsen, E. Ott, and S. M. Anlage, Phys. Rev. B **74**, 195326 (2006).
[44] R. Schäfer, H.-J. Stöckmann, T. Gorin, and T. H. Seligman, Phys. Rev. Lett. **95**, 184102 (2005).
[45] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).
[46] H. G. Winful, Phys. Rev. Lett. **91**, 260401 (2003).
[47] N. G. Kelkar, Phys. Rev. Lett. **99**, 210403 (2007).